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LINEAR LUMPED PARAMETER ANALYSIS OF SYNCHROS VI

Trigonometric Identities Useful in Synchro Analysis

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U. S. NAVAL ORDNANCE LABORATORY
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ABSTRACT: Many of the impedances in a synchro are functions of angle. An analysis of synchros will therefore involve trigonometric relations, which usually simplify greatly because of the symmetries involved. This report contains tabulated the most useful of these identities as well as derivations and suggestions for derivations.

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The linear theory of synchro analysis is being investigated under the authorization of Task NOL-Re4a-78-1-52; Fire Control Transmitting and Computing Components. The analysis, thus far presented in other papers of this series, involves many trigonometric relations which under suitable manipulations assume convenient form. It is the intent of this report to derive and tabulate the most useful of the identities which occur in synchro analysis.

The relations presented herein will find use not only in the form of analysis adopted in this series of reports, but in any type of theoretical work relating to synchros.

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A. INTRODUCTION

1. Because of the 120° symmetry prevailing in synchros, many trigonometric expressions arising from their analysis reduce to a relatively simple form, although in original appearance they may be ponderous and complicated. It is the purpose of this report to tabulate the more frequently encountered identities and to indicate methods of manipulation that are particularly useful in deriving them. The point of view adopted is to make the compilation a practical one - i.e., to include those forms that arise in practice, so that by reference to these tables the identities can be used without requiring rederivation each time. For completeness some elementary formulae are also included. Probably the best way to take advantage of this report is first to make a hasty survey to become acquainted with its contents.

B. NOTATION

2. To begin with we introduce the following notation:

$$\begin{aligned} e^{j\theta} &= \cos\theta + j\sin\theta \\ \bar{\theta} &= \theta + 120^\circ \\ \underline{\theta} &= \theta - 120^\circ \end{aligned} \qquad (1)$$

This process of using bars will apply to Greek letters θ, ϕ, . . . only.

3. We define an operator Γ to be given by

$$\Gamma\, g(\theta_1,\theta_2,\ldots\theta_n) = g(\theta_1,\theta_2,\ldots\theta_n) + g(\underline{\theta}_1,\underline{\theta}_2,\ldots\underline{\theta}_n) + g(\bar{\theta}_1,\bar{\theta}_2,\ldots\bar{\theta}_n) \qquad (2)$$

where in all cases, the bars are considered to operate on each of the variables θ_k. For example

$$\Gamma \sin\theta = \sin\theta + \sin\underline{\theta} + \sin\bar{\theta}$$

$$\Gamma \sin 2\phi = \sin 2\phi + \sin 2\underline{\phi} + \sin 2\bar{\phi}$$

$$\Gamma \sin\theta \sin 2\phi = \sin\theta \sin 2\phi + \sin\underline{\theta} \sin 2\underline{\phi} + \sin\bar{\theta} \sin 2\bar{\phi} .$$

It is evident, by the last two examples, that when a function of $n\theta$ is involved, the proper use of bars is $n\underline{\theta}$ and $n\bar{\theta}$, since bars are only applied to Greek letters.

We will also deal with functions of the type

$$A_1 g(\theta_1, \theta_2, \ldots) + A_2 g(\underline{\theta}_1, \underline{\theta}_2, \ldots) + A_3 g(\bar{\theta}_1, \bar{\theta}_2, \ldots),$$

where A_1, A_2, A_3 are constants, not necessarily equal. Here we will use a similar notation

$$\Gamma A_m g(\theta_1, \theta_2, \ldots) = A_1 g(\theta_1, \theta_2, \ldots) + A_2 g(\underline{\theta}_1, \underline{\theta}_2, \ldots) \qquad (3)$$

$$+ A_3 g(\bar{\theta}_1, \bar{\theta}_2, \ldots) .$$

In other words, if Γ operates on a function which explicitly exhibits a constant as a coefficient, it implies that in the expansion each of the terms includes coefficients which are not necessarily equal. An example is

$$\Gamma A_k \sin\theta \cos 2\phi = A_1 \sin\theta \cos 2\phi + A_2 \sin\underline{\theta} \cos 2\underline{\phi} + A_3 \sin\bar{\theta} \cos 2\bar{\phi} .$$

C. DERIVATIONS

4. The first expressions, which are cornerstones for the derivation of all succeeding formulae are

$$\Gamma \sin n\theta = \begin{cases} 0 & (n \neq 3k) \\ 3 \sin n\theta & (n = 3k) \end{cases} ,$$

$$\Gamma \cos n\theta = \begin{cases} 0 & (n \neq 3k) \\ 3 \cos n\theta & (n = 3k) \end{cases} .$$

These are derived with the aid of (1) in the following manner:

$$\begin{aligned} \Gamma \cos n\theta &= \mathrm{Re}\,(\Gamma e^{jn\theta}) \\ \Gamma \sin n\theta &= \mathrm{Im}\,(\Gamma e^{jn\theta}) \end{aligned} \tag{4}$$

where

$$\mathrm{Re}\,(A) = \text{real part of } A$$

$$\mathrm{Im}\,(A) = \text{imaginary part of } A\ .$$

$$\begin{aligned} \Gamma e^{jn\theta} &= e^{jn\theta} + e^{jn(\theta + 2\pi/3)} + e^{jn(\theta - 2\pi/3)} \\ &= e^{jn\theta}\,(1 + e^{2j\pi n/3} + e^{-2j\pi n/3}) \\ &= e^{jn\theta}\,(1 + 2\cos 2\pi n/3) \\ &= \begin{cases} 0 & (n \neq 3k) \\ 3e^{jn\theta} & (n = 3k)\ . \end{cases} \end{aligned}$$

Therefore, from (4)

$$1)\ \Gamma \cos n\theta = \begin{cases} 0 & (n \neq 3k) \\ 3\cos n\theta & (n = 3k) \end{cases}$$

$$2)\ \Gamma \sin n\theta = \begin{cases} 0 & (n \neq 3k) \\ 3\sin n\theta & (n = 3k)\ . \end{cases}$$

5. In all work relating to the product of trigonometric functions, it is helpful to transform the products to sums by means of the following identities:

$$3)\quad \sin\theta \sin\phi = (1/2)\cos(\theta - \phi) - (1/2)\cos(\theta + \phi)$$

$$4)\quad \cos\theta \cos\phi = (1/2)\cos(\theta - \phi) + (1/2)\cos(\theta + \phi)$$

$$5)\quad \sin\theta \cos\phi = (1/2)\sin(\theta - \phi) + (1/2)\sin(\theta + \phi)\ .$$

For example, substituting $\theta = \phi$ in 3) and 4) yields

$$3a) \quad \sin^2\theta = (1/2)(1 - \cos 2\theta)$$

$$4a) \quad \cos^2\theta = (1/2)(1 + \cos 2\theta) \ .$$

6. A set of identities which constantly recur in synchro analysis consists of the following:

$$6) \ \Gamma \sin^2 n\theta = \begin{cases} 3/2 & (n \neq 3k) \\ 3 \sin^2 n\theta & (n = 3k) \end{cases}$$

$$7) \ \Gamma \cos^2 n\theta = \begin{cases} 3/2 & (n \neq 3k) \\ 3 \cos^2 n\theta & (n = 3k) \end{cases}$$

$$8) \ \Gamma \sin n\theta \cos n\theta = \begin{cases} 0 & (n \neq 3k) \\ 3 \sin n\theta \cos n\theta & (n = 3k) \ . \end{cases}$$

These are easily derived using 3a) and 4a). For example,

$$\begin{aligned} \Gamma \cos^2 n\theta &= \Gamma (1/2)(1 + \cos 2n\theta) \\ &= \Gamma (1/2) + (1/2) \Gamma \cos 2n\theta \\ &= \begin{cases} 3/2 & (n \neq 3k) \\ 3 \cos^2 n\theta & (n = 3k) \ . \end{cases} \end{aligned}$$

7. Two identities which prove to be very useful are

$$9) \quad \sin n(\theta + 240°) = \sin n(\theta - 120°)$$

$$10) \quad \cos n(\theta + 240°) = \cos n(\theta - 120°) \ .$$

These are simple consequences of the fact that adding 360° or any integral multiple of 360° to the argument of a trigonometric function does not change the value of the function. Thus

$$\sin n(\theta - 120°) = \sin n(\theta - 120° + 360°) = \sin n(\theta + 240°)$$

as stated.

From 9) and 10) it follows that

$$11)\ \Gamma \sin 3n\theta \cos m\phi = \begin{cases} 0 & (m \neq 3k) \\ 3\sin 3n\theta \cos m\phi & (m = 3k) \end{cases}$$

$$12)\ \Gamma \cos 3n\theta \cos m\phi = \begin{cases} 0 & (m \neq 3k) \\ 3\cos 3n\theta \cos m\phi & (m = 3k) \end{cases}$$

$$13)\ \Gamma \sin 3n\theta \sin m\phi = \begin{cases} 0 & (m \neq 3k) \\ 3\sin 3n\theta \sin m\phi & (m = 3k) \end{cases}$$

$$14)\ \Gamma \cos 3n\theta \sin m\phi = \begin{cases} 0 & (m \neq 3k) \\ 3\cos 3n\theta \sin m\phi & (m = 3k) \end{cases}$$

since

$$3n\bar{\theta} = n(3\theta + 360°)$$

$$3n\underline{\theta} = n(3\theta - 360°) \ .$$

8. Up till now we have been considering the operator Γ as defined in (2). The study of unbalanced synchros requires the operator Γ as defined in (3). For example,

$$\Gamma A_m \cos n\theta = A_1 \cos n\theta + A_2 \cos n\underline{\theta} + A_3 \cos n\bar{\theta}$$

or $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (5)

$$\Gamma A_m \sin n\theta = A_1 \sin n\theta + A_2 \sin n\underline{\theta} + A_3 \sin n\bar{\theta}$$

where the A's are constants.

When $A_1=A_2=A_3$ these expressions simplify through the use of 1) and 2). For the case where they are not equal, we make use of the following notation:

$$A_2 = A_1 + \underline{\Delta}$$

$$A_3 = A_1 + \bar{\Delta}$$

$$A = A_1 \ .$$

Then, for example, the second form of (5) is simply

$$\sum A_m \sin n\theta = A \sum \sin n\theta + \underline{\Delta} \sin n\underline{\theta} + \bar{\Delta} \sin n\bar{\theta}$$

which, by expanding $n\underline{\theta}$ and $n\bar{\theta}$, can be written as

$$A \sum \sin n\theta + \underline{\Delta}\left[\sin n\theta \cos n(120°) - \cos n\theta \sin n(120°)\right]$$
$$+ \bar{\Delta}\left[\sin n\theta \cos n(120°) + \cos n\theta \sin n(120°)\right]. \qquad (6)$$

We now define the following notation:

$$\alpha = (1/2)(\underline{\Delta} + \bar{\Delta})$$

$$\beta = (1/2)(\underline{\Delta} - \bar{\Delta}).$$

When substituted into (6) this gives

$$23) \quad \sum A_m \sin n\theta = \begin{cases} (3A + 2\alpha) \sin n\theta & (n = 3k) \\ -\alpha \sin n\theta - \sqrt{3}\, \beta \cos n\theta & (n = 3k+1) \\ -\alpha \sin n\theta + \sqrt{3}\, \beta \cos n\theta & (n = 3k+2). \end{cases}$$

Similarly we have

$$24) \quad \sum A_m \cos n\theta = \begin{cases} (3A + 2\alpha) \cos n\theta & (n = 3k) \\ -\alpha \cos n\theta - \sqrt{3}\, \beta \sin n\theta & n = 3k+1) \\ -\alpha \cos n\theta + \sqrt{3}\, \beta \sin n\theta & (n = 3k+2). \end{cases}$$

It will be noticed that the equations 23) and 24) are linear combinations of $\sin n\theta$ and $\cos n\theta$. These combinations can be put into the handy form of $C\left[\sin (n\theta - \delta)\right]$ where C and δ are constants. Let

$$A \sin n\theta + B \cos n\theta = C \sin (n\theta - \delta)$$
$$= C\left[\sin n\theta \cos \delta\right] - C\left[\sin \delta \cos n\theta\right].$$

Equating the coefficients of $\sin n\theta$ and $\cos n\theta$ we find

$$A = C \cos \delta$$
$$B = -C \sin \delta.$$

Consequently δ is found from

$$\tan \delta = -B/A$$

and C is given by

$$C = \sqrt{A^2 + B^2} \quad .$$

9. An important special case of the identities treated in paragraph 8 is that of

$$\mu = A_1 \sin 2\theta + A_2 \sin 2\underline{\theta} + A_3 \sin 2\bar{\theta} \qquad (7)$$

which is equivalent to

$$\mu = B \sin (2\theta - \delta) \quad . \qquad (8)$$

Expanding (7) we obtain

$$\mu = (1/2)(2\,A_1 - A_2 - A_3)\sin 2\theta + (\sqrt{3}/2)(A_2 - A_3)\cos 2\theta \ . \qquad (9)$$

Equating (8) and (9) we find

$$\tan \delta = (\sqrt{3})(A_2 - A_3)/(2A_1 - A_2 - A_3)$$

and

$$B = \sqrt{A_1^2 + A_2^2 + A_3^2 - A_1A_2 - A_1A_3 - A_2A_3} \quad .$$

10. The next set of identities we treat, which pertain especially to unbalanced synchros, are of the form

$$A_1 \sin n\theta \cos \theta + A_2 \sin n\underline{\theta} \cos \underline{\theta} + A_3 \sin n\bar{\theta} \cos \bar{\theta}$$

or

$$A_1 \cos n\theta \cos 2\theta + A_2 \cos n\underline{\theta} \cos 2\underline{\theta} + A_3 \cos n\bar{\theta} \cos 2\bar{\theta}$$

and other similar combinations which interchange sin and cos. The A's are constants.

11. When $A_1 = A_2 = A_3$ these expressions can be evaluated by the use of Table 4. When this condition does not hold we expand the expressions and combine the resulting terms.

Thus

$$A_1 \sin n\theta \cos\theta + A_2 \sin n\underline{\theta} \cos\underline{\theta} + A_3 \sin n\bar{\theta} \cos\bar{\theta}$$

$$= A\,\Gamma\left[\sin n\theta \cos\theta + \underline{\Delta}\sin n\theta \cos\theta + \bar{\Delta}\sin n\bar{\theta}\cos\bar{\theta}\right.$$

$$= A\,\Gamma\Big[\sin n\theta \cos\theta + (\underline{\Delta}/2)\left[\sin(n-1)\underline{\theta} + \sin(n+1)\underline{\theta}\right]$$

$$+ (\bar{\Delta}/2)\left[\sin(n-1)\bar{\theta} + \sin(n+1)\bar{\theta}\right]$$

$$= A\,\Gamma\Big[\sin n\theta \cos\theta + \alpha \sin(n-1)\theta \cos(n-1)120^\circ + \beta \cos(n-1)\theta \sin(n-1)120^\circ$$

$$+ \alpha \sin(n+1)\theta \cos(n+1)120^\circ + \beta \cos(n+1)\theta \sin(n+1)120^\circ \,. \qquad (10)$$

A complete list of such expressions is tabulated in Tables V and VI.

D. TABLES AND FORMULAE

12. The important identities can be presented in tabular form. Table I is a summary of the relations previously derived, as well as certain other relations which follow by similar processes. Tables II, III, and IV present the results of applying the operator Γ as defined in (2) on specific trigonometric functions which arise in the analysis of balanced synchros. Tables V and VI apply to the operator Γ when defined as in (3) and are particularly useful for the analysis of unbalanced synchros.

13. A few examples will now be given in order to familiarize the reader with the use of the tables.

Example 1. Find $\Gamma \cos\theta \cos\phi$.

Look in Table II in the first row and column and find

$$\Gamma \cos\theta \cos\phi = (3/2)\cos(\theta - \phi) \,.$$

Example 2. Find $\cos\theta \cos\underline{\phi} + \cos\underline{\theta}\cos\bar{\phi} + \cos\bar{\theta}\cos\phi$.

Substitute $\phi = \underline{\phi}$ in example 1 to find

$$\Gamma \cos\theta \cos\underline{\phi} = (3/2)\cos(\theta - \underline{\phi}) = (3/2)\cos(\overline{\theta - \phi}) \,.$$

Example 3. Find $\Gamma \sin 2\phi \sin^2\phi$.

In Table III, with $\phi = \theta$, we find $\Gamma \sin 2\phi \sin^2\phi = 0$.

Example 4. Find the value of

$$\cos\theta\cos\underline{\theta}\cos 2\bar{\theta} + \cos\underline{\theta}\cos\bar{\theta}\cos 2\theta + \cos\bar{\theta}\cos\theta\cos 2\underline{\theta} \;.$$

From 15), with θ substituted for ϕ, we see that this expression is equal to $3/4$.

Example 5. Find $\Gamma \sin 7\theta \sin 2\theta$.

With $n = 2$, we find from Table IV that $\Gamma \sin 7\theta \sin 2\theta = -(3/2) \cos 9\theta$.

Example 6. Find $\sin 4\theta \sin\theta + 3 \sin 4\underline{\theta} \sin\underline{\theta} + 5 \sin 4\bar{\theta} \sin\bar{\theta} = p$.

From the given expression, we find $\underline{\Delta} = 2$, $\bar{\Delta} = 4$. Therefore $\alpha = 3$, $\beta = -1$. From Table Va, noting that $n = 4$ is of the form $n = 3k+1$, we find

$$p = (3/2)\cos(n+1)\theta + (\sqrt{3}/2)\sin(n+1)\theta + (9/2)\cos(n-1)\theta$$

$$= (3/2)\cos 5\theta + (\sqrt{3}/2)\sin 5\theta + (9/2)\cos 3\theta \;.$$

TABLE I

List of Formulae

$$1)\quad \Gamma \cos n\theta = \begin{cases} 0 & (n \neq 3k) \\ 3 \cos n\theta & (n = 3k) \end{cases}$$

$$2)\quad \Gamma \sin n\theta = \begin{cases} 0 & (n \neq 3k) \\ 3 \sin n\theta & (n = 3k) \end{cases}$$

$$3)\quad \sin\theta \sin\phi = (1/2)\cos(\theta - \phi) - (1/2)\cos(\theta + \phi)$$

$$4)\quad \cos\theta \cos\phi = (1/2)\cos(\theta - \phi) + (1/2)\cos(\theta + \phi)$$

$$5)\quad \cos\theta \sin\phi = (1/2)\sin(\phi - \theta) + (1/2)\sin(\theta + \phi)$$

$$3a)\quad \sin^2\theta = (1/2)(1 - \cos 2\theta)$$

$$4a)\quad \cos^2\theta = (1/2)(1 + \cos 2\theta)$$

$$6)\quad \Gamma \sin^2 n\theta = \begin{cases} 3/2 & (n \neq 3k) \\ 3 \sin^2 n\theta & (n = 3k) \end{cases}$$

$$7)\quad \Gamma \cos^2 n\theta = \begin{cases} 3/2 & (n \neq 3k) \\ 3 \cos^2 n\theta & (n = 3k) \end{cases}$$

$$8)\quad \sin n\theta \cos n\theta = \begin{cases} 0 & (n \neq 3k) \\ 3 \sin n\theta \cos n\theta & (n = 3k) \end{cases}$$

$$9)\quad \sin n(\theta + 240^\circ) = \sin n(\theta - 120^\circ)$$

$$10)\quad \cos n(\theta + 240^\circ) = \cos n(\theta - 120^\circ)$$

$$11)\quad \Gamma \sin 3n\theta \cos m\phi = \begin{cases} 0 & (m \neq 3k) \\ 3 \sin 3n\theta \cos m\phi & (m = 3k) \end{cases}$$

$$12)\quad \Gamma \cos 3n\theta \cos m\phi = \begin{cases} 0 & (m \neq 3k) \\ 3 \cos 3n\theta \cos m\phi & (m = 3k) \end{cases}$$

$$13)\quad \Gamma \sin 3n\theta \sin m\phi = \begin{cases} 0 & (m \neq 3k) \\ 3 \sin 3n\theta \sin m\phi & (m = 3k) \end{cases}$$

$$14)\quad \Gamma \cos 3n\theta \sin m\phi = \begin{cases} 0 & (m \neq 3k) \\ 3 \cos 3n\theta \sin m\phi & (m = 3k) \end{cases}$$

$$15)\quad \cos\theta \cos\underline{\theta} \cos 2\bar{\phi} + \cos\underline{\theta} \cos\bar{\theta} \cos 2\phi + \cos\theta \cos\bar{\theta} \cos 2\underline{\phi}$$
$$= (3/4)\cos 2(\theta - \phi)$$

$$16)\quad \sin\theta \sin\underline{\theta} \cos 2\bar{\phi} + \sin\underline{\theta} \sin\bar{\theta} \cos 2\phi + \sin\theta \sin\bar{\theta} \cos 2\underline{\phi}$$
$$= -(3/4)\cos 2(\theta - \phi)$$

$$17)\quad \sin\theta \sin\underline{\theta} \sin 2\bar{\phi} + \sin\underline{\theta} \sin\bar{\theta} \sin 2\phi + \cos\theta \cos\bar{\theta} \sin 2\underline{\phi}$$
$$= (3/4)\sin 2(\theta - \phi)$$

$$18)\quad \cos\theta \cos\underline{\theta} \sin 2\bar{\phi} + \cos\underline{\theta} \cos\bar{\theta} \sin 2\phi + \cos\theta \cos\bar{\theta} \sin 2\underline{\phi}$$
$$= -(3/4)\sin 2(\theta - \phi)$$

$$19)\quad \sin\theta \pm \sin\phi = 2 \sin[(1/2)(\theta \pm \phi)] \cos[(1/2)(\theta \mp \phi)]$$

$$20)\quad \cos\theta + \cos\phi = 2 \cos[(1/2)(\theta + \phi)] \cos[(1/2)(\theta - \phi)]$$

$$21)\quad \cos\theta - \cos\phi = -2 \sin[(1/2)(\theta + \phi)] \sin[(1/2)(\theta - \phi)]$$

$$22)\quad \begin{vmatrix} \cos\theta & \sin\theta \\ \cos\underline{\theta} & \sin\underline{\theta} \end{vmatrix} = -(\sqrt{3}/2)$$

$$23)\quad \Gamma A_m \sin n\theta = \begin{cases} (3A + 2\alpha) \sin n\theta & (n = 3k) \\ -\alpha \sin n\theta - \sqrt{3}\,\beta \cos n\theta & (n = 3k+1) \\ -\alpha \sin n\theta + \sqrt{3}\,\beta \cos n\theta & (n = 3k+2) \end{cases}$$

$$24)\quad \Gamma A_m \cos n\theta = \begin{cases} (3A + 2\alpha) \cos n\theta & (n = 3k) \\ -\alpha \cos n\theta - \sqrt{3}\,\beta \sin n\theta & (n = 3k+1) \\ -\alpha \cos n\theta + \sqrt{3}\,\beta \sin n\theta & (n = 3k+2) \end{cases}$$

(see next page for definition of α, β, A)

where

$$\alpha = (1/2)(\underline{\Delta}+\bar{\Delta})$$

$$\beta = (1/2)(\underline{\Delta}-\bar{\Delta})$$

$$A_2 = A + \underline{\Delta}$$

$$A_3 = A + \bar{\Delta}$$

$$A_1 = A$$

25) $$\Gamma A_m \sin 2\theta = B \sin(2\theta - \delta)$$

$$B = \sqrt{A_1^2 + A_2^2 + A_3^2 - A_1A_2 - A_1A_3 - A_2A_3}$$

$$\tan\delta = \sqrt{3}(A_2 - A_3)/(2A_1 - A_2 - A_3)$$

26) $$\Gamma \cos 2\phi \cos\theta \cos\lambda = (3/4)\cos 2[\phi - (\theta+\lambda)/2]$$

27) $$\Gamma \cos 2\phi \cos\theta \sin\lambda = (3/4)\sin 2[\phi - (\theta+\lambda)/2]$$

28) $$\Gamma \cos 2\phi \sin\theta \sin\lambda = -(3/4)\cos 2[\phi - (\theta+\lambda)/2]$$

29) $$\begin{aligned} &\cos 2\phi\,(\cos\underline{\theta}\cos\bar{\lambda} + \cos\bar{\theta}\cos\underline{\lambda}) \\ &+ \cos 2\underline{\phi}\,(\cos\bar{\theta}\cos\lambda + \cos\theta\cos\bar{\lambda}) \\ &+ \cos 2\bar{\phi}\,(\cos\theta\cos\underline{\lambda} + \cos\underline{\theta}\cos\lambda) \\ &= (3/2)\cos 2[\phi - (\theta+\lambda)/2] \end{aligned}$$

30) $$\begin{aligned} &\cos 2\phi\,(\sin\underline{\theta}\sin\bar{\lambda} + \sin\bar{\theta}\sin\underline{\lambda}) \\ &+ \cos 2\underline{\phi}\,(\sin\theta\sin\bar{\lambda} + \sin\bar{\theta}\sin\lambda) \\ &+ \cos 2\bar{\phi}\,(\sin\theta\sin\underline{\lambda} + \sin\underline{\theta}\sin\lambda) \\ &= -(3/2)\cos 2[\phi - (\theta+\lambda)/2] \end{aligned}$$

31) $$\begin{aligned} &\cos 2\phi\,(\sin\underline{\theta}\cos\bar{\lambda} + \sin\bar{\theta}\cos\underline{\lambda}) \\ &+ \cos 2\underline{\phi}\,(\sin\theta\cos\bar{\lambda} + \sin\bar{\theta}\cos\lambda) \\ &+ \cos 2\bar{\phi}\,(\sin\theta\cos\underline{\lambda} + \sin\underline{\theta}\cos\lambda) \\ &= -(3/2)\sin 2[\phi - (\theta+\lambda)/2] \end{aligned}$$

32) $$\begin{bmatrix} \cos\phi & \sin\phi \\ \cos\underline{\phi} & \sin\underline{\phi} \\ \cos\bar{\phi} & \sin\bar{\phi} \end{bmatrix} \cdot \begin{bmatrix} \cos(\theta-\phi) & \sin(\theta-\phi) \\ -\sin(\theta-\phi) & \cos(\theta-\phi) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \cos\underline{\theta} & \sin\underline{\theta} \\ \cos\bar{\theta} & \sin\bar{\theta} \end{bmatrix}$$

33) $$\begin{bmatrix} \cos\theta & \sin\theta \\ \cos\underline{\theta} & \sin\underline{\theta} \\ \cos\bar{\theta} & \sin\bar{\theta} \end{bmatrix} \cdot \begin{bmatrix} \cos(\theta-\phi) & -\sin(\theta-\phi) \\ \sin(\theta-\phi) & \cos(\theta-\phi) \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ \cos\underline{\phi} & \sin\underline{\phi} \\ \cos\bar{\phi} & \sin\bar{\phi} \end{bmatrix}$$

34) $$\begin{bmatrix} -\sin\theta & \cos\theta \\ -\sin\underline{\theta} & \cos\underline{\theta} \\ -\sin\bar{\theta} & \cos\bar{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \cos\underline{\theta} & \sin\underline{\theta} \\ \cos\bar{\theta} & \sin\bar{\theta} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

35) Let $\sigma = \theta - \phi - \pi/2$. Then

$$\begin{bmatrix} \cos(\theta-\phi) & \sin(\theta-\phi) \\ -\sin(\theta-\phi) & \cos(\theta-\phi) \end{bmatrix} = \begin{bmatrix} -\sin\sigma & \cos\sigma \\ -\cos\sigma & -\sin\sigma \end{bmatrix}$$

TABLE II. $\Gamma f(\phi)g(\theta)$

$f(\phi)$ \ $g(\theta)$	$(2/3)\cos\theta$	$(2/3)\sin\theta$	$(2/3)\cos 2\theta$	$(2/3)\sin 2\theta$
$\cos\phi$	$\cos(\theta-\phi)$	$\sin(\theta-\phi)$	$\cos(2\theta+\phi)$	$\sin(2\theta+\phi)$
$\sin\phi$	$-\sin(\theta-\phi)$	$\cos(\theta-\phi)$	$\sin(2\theta+\phi)$	$-\cos(2\theta+\phi)$
$\cos 2\phi$	$\cos(2\phi+\theta)$	$\sin(2\phi+\theta)$	$\cos 2(\theta-\phi)$	$\sin 2(\theta-\phi)$
$\sin 2\phi$	$\sin(2\phi+\theta)$	$-\cos(2\phi+\theta)$	$-\sin 2(\theta-\phi)$	$\cos 2(\theta-\phi)$

TABLE III. $\Gamma f(\phi)g(\theta)$

$f(\phi)$ \ $g(\theta)$	$(4/3)\sin^2\theta$	$(4/3)\cos^2\theta$
$\sin\phi$	$-\sin(2\theta+\phi)$	$\sin(2\theta+\phi)$
$\cos\phi$	$-\cos(2\theta+\phi)$	$\cos(2\theta+\phi)$
$\sin 2\phi$	$\sin 2(\theta-\phi)$	$-\sin 2(\theta-\phi)$
$\cos 2\phi$	$-\cos 2(\theta-\phi)$	$\cos 2(\theta-\phi)$

<u>How to read Tables II and III.</u>

The value in each box is $\Gamma f(\phi)\, g(\theta)$ where $f(\phi)$ and $g(\theta)$ are the headings of the rows and columns respectively to which the box belongs.

For example, in Table II,

$$\Gamma \cos 2\phi \cos\theta = (3/2)\cos(2\phi+\theta) .$$

TABLE* IV. $\Gamma\ f(\theta)g(\theta)$

$f(\theta)$ \ $g(\theta)$	$(2/3)\cos\theta$	$(2/3)\sin\theta$	$(2/3)\cos 2\theta$	$(2/3)\sin 2\theta$	K = Constant
$\cos(3n+1)\theta$	$\cos 3n\theta$	$-\sin 3n\theta$	$\cos 3(n+1)\theta$	$\sin 3(n+1)\theta$	0
$\cos(3n+2)\theta$	$\cos 3(n+1)\theta$	$\sin 3(n+1)\theta$	$\cos 3n\theta$	$-\sin 3n\theta$	0
$\cos 3n\theta$	0	0	0	0	$3K\cos 3n\theta$
$\sin(3n+1)\theta$	$\sin 3n\theta$	$\cos 3n\theta$	$\sin 3(n+1)\theta$	$-\cos 3(n+1)\theta$	0
$\sin(3n+2)\theta$	$\sin 3(n+1)\theta$	$-\cos 3(n+1)\theta$	$\sin 3n\theta$	$\cos 3n\theta$	0
$\sin 3n\theta$	0	0	0	0	$3K\sin 3n\theta$

* Reprinted from Navord 1710, "Linear Lumped Parameter Analysis of Synchros I" by J. H. Rosenbloom, p.28

<u>How to read Table IV.</u>

Read the same as Tables II and III.

TABLE Va): $\Gamma A_m g(n\theta)\, f(\theta)$

	$\cos(n+1)\,\theta$	$\sin(n+1)\,\theta$	$\cos(n-1)\,\theta$	$\sin(n-1)\,\theta$
$\Gamma A_m \cos n\theta \cos\theta$ $n = 3K$	$-\frac{1}{2}\alpha$	$-\frac{\sqrt{3}}{2}\beta$	$-\frac{1}{2}\alpha$	$\frac{\sqrt{3}}{2}\beta$
$n = 3K+1$	$-\frac{1}{2}\alpha$	$\frac{\sqrt{3}}{2}\beta$	α	$-\frac{3}{2}A$
$n = 3K+2$	α	$\frac{3}{2}A$	$-\frac{1}{2}\alpha$	$-\frac{\sqrt{3}}{2}\beta$
$\Gamma A_m \sin n\theta \sin\theta$ $n = 3K$	$\frac{1}{2}\alpha$	$\frac{\sqrt{3}}{2}\beta$	$-\frac{1}{2}\alpha$	$\frac{\sqrt{3}}{2}\beta$
$n = 3K+1$	$\frac{1}{2}\alpha$	$-\frac{\sqrt{3}}{2}\beta$	$\frac{3}{2}A + \alpha$	0
$n = 3K+2$	$-\alpha$	0	$\frac{3}{2}A - \frac{1}{2}\alpha$	$-\frac{\sqrt{3}}{2}\beta$

<u>How to read Table Va.</u>

See Page 20.

TABLE Vb): $\Gamma A_m g(n\theta)\, f(\theta)$

	$\cos(n+1)\theta$	$\sin(n+1)\theta$	$\cos(n-1)\theta$	$\sin(n-1)\theta$
$\Gamma A_m \sin n\theta \cos\theta$ $n = 3K$	$\frac{\sqrt{3}}{2}\beta$	$-\frac{1}{2}\alpha$	$-\frac{\sqrt{3}}{2}\beta$	$-\frac{1}{2}\alpha$
$n = 3K+1$	$-\frac{\sqrt{3}}{2}\beta$	$-\frac{1}{2}\alpha$	0	$\frac{3}{2}A + \alpha$
$n = 3K+2$	0	$\frac{3}{2}A + \alpha$	$\frac{\sqrt{3}}{2}\beta$	$-\frac{1}{2}\alpha$
$\Gamma A_m \cos n\theta \sin\theta$ $n = 3K$	$\frac{\sqrt{3}}{2}\beta$	$-\frac{1}{2}\alpha$	$\frac{\sqrt{3}}{2}\beta$	$\frac{1}{2}\alpha$
$n = 3K+1$	$-\frac{\sqrt{3}}{2}\beta$	$-\frac{1}{2}\alpha$	$\frac{3}{2}A$	$-\alpha$
$n = 3K+2$	$\frac{3}{2}A$	α	$-\frac{\sqrt{3}}{2}\beta$	$\frac{1}{2}\alpha$

How to read Table Vb.

See Page 20.

TABLE VIa): $\Gamma A_m g(n\theta)\, f(2\theta)$

	$\cos(n+2)\theta$	$\sin(n+2)\theta$	$\cos(n-2)\theta$	$\sin(n-2)\theta$
$\Gamma A_m \cos n\theta \cos 2\theta$ $n = 3K$	$-\frac{1}{2}\alpha$	$\frac{\sqrt{3}}{2}\beta$	$-\frac{1}{2}\alpha$	$-\frac{\sqrt{3}}{2}\beta$
$n = 3K+1$	$\frac{3}{2}A+\alpha$	0	$-\frac{1}{2}\alpha$	$\frac{\sqrt{3}}{2}\beta$
$n = 3K+2$	$-\frac{1}{2}\alpha$	$\frac{3}{2}A-\frac{\sqrt{3}}{2}\beta$	α	0
$\Gamma A_m \sin n\theta \sin 2\theta$ $n = 3K$	$\frac{1}{2}\alpha$	$-\frac{\sqrt{3}}{2}\beta$	$-\frac{1}{2}\alpha$	$-\frac{\sqrt{3}}{2}\beta$
$n = 3K+1$	$-\frac{3}{2}A-\alpha$	0	$-\frac{1}{2}\alpha$	$\frac{\sqrt{3}}{2}\beta$
$n = 3K+2$	$\frac{1}{2}\alpha$	$\frac{\sqrt{3}}{2}\beta$	$\frac{3}{2}A+\alpha$	0

How to read Table VIa.

See Page 20.

TABLE VIb): $\Gamma A_m g(n\theta)\, f(2\theta)$

	$\cos(n+2)\,\theta$	$\sin(n+2)\,\theta$	$\cos(n-2)\,\theta$	$\sin(n-2)\,\theta$
$\Gamma A_m \cos n\theta \sin 2\theta$ $n = 3K$	$-\frac{\sqrt{3}}{2}\beta$	$-\frac{1}{2}\alpha$	$-\frac{\sqrt{3}}{2}\beta$	$\frac{1}{2}\alpha$
$n = 3K+1$	0	$\frac{3}{2}A+\alpha$	$\frac{\sqrt{3}}{2}\beta$	$\frac{1}{2}\alpha$
$n = 3K+2$	$\frac{\sqrt{3}}{2}\beta$	$-\frac{1}{2}\alpha$	0	$-\frac{3}{2}A-\alpha$
$\Gamma A_m \sin n\theta \cos 2\theta$ $n = 3K$	$-\frac{\sqrt{3}}{2}\beta$	$-\frac{1}{2}\alpha$	$\frac{\sqrt{3}}{2}\beta$	$-\frac{1}{2}\alpha$
$n = 3K+1$	0	$\frac{3}{2}A+\alpha$	$-\frac{\sqrt{3}}{2}\beta$	$-\frac{1}{2}\alpha$
$n = 3K+2$	$\frac{\sqrt{3}}{2}\beta$	$-\frac{1}{2}\alpha$	0	$\frac{3}{2}A+\alpha$

<u>How to read Table VIb.</u>

See Page 20.

How to read Tables V and VI.

Compute α, β and A from

$$\underline{\Delta} = A_2 - A_1$$

$$\ddot{\Delta} = A_3 - A_1$$

$$A = A_1$$

$$\alpha = (1/2)(\underline{\Delta} + \ddot{\Delta})$$

$$\beta = (1/2)(\underline{\Delta} - \ddot{\Delta})$$

$$\Gamma A_m \, g(\theta) = A_1 \, g(\theta) + A_2 \, g(\underline{\theta}) + A_3 \, g(\ddot{\theta})$$

Consider a typical row

	cos (n+1)	sin (n+1)	cos (n-1)	sin (n-1)
$\Gamma A_m \, g(\theta) \cos \theta$	b_1	b_2	b_3	b_4

Then

$$\Gamma A_m \, g(\theta) \cos \theta = b_1 \cos(n+1)\theta + b_2 \sin(n+1)\theta + b_3 \cos(n-1)\theta$$
$$+ b_4 \sin(n-1)\theta$$

For example, from Table Va,

$$\Gamma A_m \cos n\theta \cos \theta = -(1/2)\alpha \cos (n+1)\theta + (\sqrt{3}/2)\beta \sin (n+1)\theta$$
$$+ \cos (n-1)\theta - (3/2) A \sin (n-1)\theta \; .$$

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